ON SHRINKING ARCS IN METRIC SPACES

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ABSTRACT

By a sin $(1/x)$-curve is meant a metric continuum that is a 1-1 continuous image of the disjoint union of an arc and a semi-open interval that has the image of the arc as continuum of convergence. It is shown that if M is a compact metric space, $A \subset M$ an arc, while M/A is an arc having A/A as an end-point, then M is an arc, a triod, some sin $(1/x)$-curve, or some sin $(1/x)$-curve with an arc attached at one point, or some sin $(1/x)$-curve with two arcs attached. The case of shrinking finitely many arcs is also considered in an attaching theorem.

If M is a compact metric space, $\mathscr{D}$ an upper semicontinuous decomposition of M, let $X = M/\mathscr{D}$ and suppose $\eta: M \to X$ is the natural map.(2) We will examine the case in which $\mathscr{D}$ has only finitely many nondegenerate elements that are arcs. For X an arc, Theorem 1 describes the topological type of M where $\mathscr{D}$ has only one arc element. In Theorem 2 this result is generalized to the case of finitely many arcs. In the sequel M/A will denote the topological space obtained from the topological space M by identifying the points of the subset A of M.

Lemma 1. *Let M be a compact metric space and $A \subset M$ a closed subset such that $X = M/A$ is an arc with $\eta(A) = \omega$ as an end-point under the natural map $\eta: M \to X$. Then $\overline{M - A} \cap A$ is a continuum.*

Proof. The map $\eta \mid M - A$ is a 1-1 map that is open and is thus a homeomorphism. It follows that $M - A$ is topologically the semiopen interval $[0, 1)$. Thus $\overline{M - A}$ is a continuum with $\overline{M - A} - (M - A) \subset A$. *So without loss of generality one may assume that $M = \overline{M - A}$ and that $A = (\overline{M - A}) \cap A$.*

Let $g: [0, 1) \to M - A$ be a homeomorphism of $[0, 1)$ onto $M - A$ and define $C_k = g\{[(k/k + 1), 1)\} \cup A$ for $k = 0, 1, 2, \cdots, n, \cdots$. Then C_k is a continuum and $\cap_1^\infty C_k = A = \overline{M - A} \cap A$ is a continuum since $C_k \supset C_{k+1}$.

Theorem 1. *M is a compact metric space and $A \subset M$ is an arc. If $X = M/A$ is an arc with $\eta(A) = \omega$ as an end-point under the natural map $\eta: M \to X$, then M is an arc, a triod, the union $S \cup K$ of a sin $(1/x)$ – curve S $(0 < x \leqq 1)$ and*

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(2) The notation $X = M/\mathscr{D}$ is not commonly used and an upper semi-continuous decomposition of a space M is usually denoted by $\mathscr{D}$. We prefer however to distinquish between the family $\mathscr{D}$ and the topological space $\mathscr{D}$ (denoted here by $X = M/\mathscr{D}$).

the interval of convergence $K = \{(x, y), x = 0, -1 \leqq y \leqq 1\}$, *the set* $S \cup K$ *with an arc attached at an end-point of the interval of convergence, or the set* $S \cup K$ *with two disjoint arcs so attached, one at each of the end-points of the interval of convergence.*(3)

Proof. By Lemma 1, $C = \overline{M - A} \cap A$ is a continuum and since A is an arc, C is a point or an arc. If C is a point, then $\overline{M - A} = (M - A) \cup C$ is the 1-point compactification of a half-open arc and is thus an arc. Hence M is a union of two arcs $\overline{M - A}$ and A with the end-point C of $\overline{M - A}$ on A. Clearly M is an arc or triod.

In case C is an arc, let its end-points be a and b. Since $M - A$ is a semi-arc and $C = \overline{M - A} \cap A$, one can represent $M - A$ as a union of arcs A_n and B_n: $M - A = \bigcup_{n=1}^{\infty}(A_n \cup B_n)$ where $A_n = [a_n, b_n]$ $B_n = [b_n, a_{n+1}]$ $n = 1, 2, \cdots$ and a_n, b_n denote the end-points of these arcs. Moreover one can assume that $A_n \cap B_n = (b_n), B_n \cap A_{n+1} = (a_{n+1})$ and $a_n \to a$ and $b_n \to b$ where a_1 is the end-point of $M - A(\eta(a_1) \neq \omega)$. Then representing similarly a $\sin 1/x$-curve $S(0 < x \leqq 1)$ one can find a mapping of the interval K onto C obtaining a homeomorphism between $S \cup K$ and $(M - A) \cup C$. It follows that $\overline{M - A}$ is topologically $S \cup K$. If $C = A$, M is (topologically) the set $S \cup K$. Otherwise there are two cases. If $C \neq A$, C either lies in the interior of A or C has an end-point in common with A. These two cases yield the last two values of M in Theorem 1.

In Theorem 1, there occur five different class counter-images of an arc. We call this set of continua F. Suppose that M is a compact metric space, $A \subset M$ is an arc while $X = M/A$ is an arc with $\eta(A) = \omega$ an interior point of X. Then X is a union of two arcs X_1 and X_2; $X_1 \cap X_2 = \omega$ is an end-point of each. From Theorem 1 we know that each of the spaces $M_i = \eta^{-1}(X_i)(i = 1, 2)$ is a member of F, and $M_1 \cap M_2 \subset A$. Thus X is obtained by attaching members of F at points of an arc.

Using the operation of attaching of two spaces by a function f one can prove the following:

THEOREM 2. *Let* M *be a compact metric space and* $\mathscr{D}$ *an upper semi-continuous decomposition of* M *whose only non-degenerate elements are a finite number of arcs* $A_1, A_2, \cdots, A_n$. *If* $X = M/\mathscr{D}$ *is an arc then* M *can be represented as a union* $M_1 \cup M_2 \cup \cdots \cup M_n$ *so that* $M_i \in F$, $i = 1, 2, \cdots, n$ *and* $M_1 \cup \cdots \cup M_{i+1}$ *is obtained from* $M_1 \cup \cdots \cup M_i$ *by attaching* M_{i+1} *to* $M_1 \cup \cdots \cup M_i$ *by a continuous function* f_i, $i = 1, 2, n - 1$.

The question what 1-dimensional continua can be decomposed into arcs and points so that the hyperspace is a preassigned continuum may be of some interest.

(3) Here $S \cup K$ denotes any metric continuum that is a 1-1 continuous image of a semi-open interval S and a disjoint arc K having K as continuum of convergence. We thank H. Davis for this definition.

It should perhaps be pointed out that a relationship exists between the observations made here and the cyclic element theory as presented in [1], [2], [3]. Though the continua we have considered here are not locally connected, Theorem 2 does provide us with a representation of a continuum analogous to the cyclic chains. Links in the chain may or may not be locally connected in our case. The development in [3] of the higher order cyclic element theory is actually carried out for compact finite dimensional metric spaces.

References

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